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COMPUTATIONAL FORMULATION FOR PERIODIC VIBRATION OF GEOMETRICALLY NONLINEAR STRUCTURES—PART 2: NUMERICAL STRATEGY AND EXAMPLES

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Abstract—In this paper the numerical strategy for solving the matrix amplitude equation with parameter is discussed in detail. The matrix amplitude equation is the result of application of the Galerkin method for analysis of periodic solutions of the geometrically nonlinear structures. A theoretical background of the proposed method of analysis is given in a companion paper by Lewandowski [Lewandowski, R. (1996). Computational formulation for periodic vibration of geometrically nonlinear structures—Part 1: theoretical background. *International Journal of Solids and Structures* **34**(15), 1925–1947]. An example application of general theory is also presented. Numerical results are given and discussed to show some interesting behaviors of nonlinear structures and to illustrate the numerical efficiency and validity of the suggested method. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

A theoretical background for nonlinear free and steady-state vibrations of geometrically nonlinear structures is presented in a companion paper by Lewandowski (1996). In this paper the numerical strategy for solving a resulting set of nonlinear algebraic equations and an example application of general theory for beams vibration are presented. Moreover, an analysis of critical points which may exist on the response curves is given. Numerical applications to interesting nonlinear beam structures are given to demonstrate the efficiency and accuracy of the method.

The most important things of the proposed method of analysis are recalled here for convenience and to make this paper more self-consistent. The motion equation of geometrically nonlinear structures can be written in the form

$$\mathbf{R}(\mathbf{u}) = \mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + [\mathbf{K}_0 + \mathbf{K}_1(\mathbf{u}) + \tfrac{1}{2}\mathbf{K}_2(\mathbf{u})]\mathbf{u} - \mathbf{P}(t) = \mathbf{0}, \tag{1}$$

where $\mathbf{u}$ and $\mathbf{P}(t)$ are the vector of nodal displacements and the vector for nodal excitation forces, respectively. Moreover, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}_0$, $\mathbf{K}_1(\mathbf{u})$ and $\mathbf{K}_2(\mathbf{u})$ are the mass matrix, the damping matrix, the linear stiffness matrix and the nonlinear matrices describing a nonlinear part of vector of nodal restoring forces, respectively. The approximate solution of motion equation, in a case of harmonically varying excitation forces with the excitation frequency λ, is assumed to be in a form of the following truncated Fourier series

$$\mathbf{u}(t) = \mathbf{a}_i \cos z_i\lambda t + \mathbf{b}_i \sin z_i\lambda t, \tag{2}$$

where z_i are integer number, $i = 1, 2, \ldots, n$ and the summation convention is adopted for repeated indices. The unknown vectors of amplitudes of harmonics $\mathbf{a}_i$, $\mathbf{b}_i$ are determined from the matrix amplitude equation. This equation can be derived by using the Galerkin method and written in the following compact form

$$\mathbf{G}(\lambda, \mathbf{a}, \mathbf{P}) = [\mathbf{H}(\mathbf{a}) + \lambda\mathbf{D} - \lambda^2\mathbf{B}]\mathbf{a} - \mathbf{P} = \mathbf{0}, \tag{3}$$

where **a** is the vector which contains the amplitudes of all harmonics. The matrices **B** and **D** are built on a basis of mass and damping matrices, respectively, and **H**(**a**) is the nonlinear matrix function of amplitudes. In a case of free vibration problem the approximate solution of motion equation is assumed in a form of truncated cosine Fourier series and the problem is completely determined after solving the following nonlinear eigenvalue problem

$$\mathbf{G}(\mathbf{a}, \omega) = [\mathbf{H}(\mathbf{a}) - \omega^2\mathbf{B}]\mathbf{a} = \mathbf{0}, \tag{4}$$

where $\omega = \lambda$ denotes the nonlinear, natural frequency.

The solution of motion equation for the chosen frequency and amplitudes of excitation forces do not fully characterize a behavior of structures. Usually, the so-called response curves, i.e. the relation between the amplitudes of vibration and the excitation or natural frequency are determined. A very typical example of backbone and response curves is shown in Fig. 1 and from this it is obvious that the critical points can occur. For these reasons, the matrix amplitude eqn (3) or (4) must be considered as an equation with parameter and the appropriate numerical procedures must be applied to enable the solution of a problem of this type.

The most powerful method of solution of the matrix equation with parameter is probably the continuation method. A theoretical basis of this method can be found in a book by Seydel (1988). In a field of nonlinear dynamics of structures also other methods like the Newton–Raphson method used by Chia (1980), the method proposed by Mei (1972) and later extended by Lewandowski (1989, 1993), the vector iteration method proposed by Lewandowski (1985, 1993) and the Berger method used by Wellford *et al.* (1980) are successfully applied to solve the nonlinear eigenvalue problem (4). However, so far these methods are applied to solve the amplitude equation when only one harmonic is taken into account in an assumed solution of free vibration problems.

In this paper only an application of the continuation method is described in detail. A review of all of previously mentioned methods is presented in paper by Lewandowski (1993).

2. SOLUTION OF MATRIX AMPLITUDE EQUATION BY CONTINUATION METHOD

2.1. *Preliminary consideration*

The matrix amplitude equation in a case of free vibration problem has the trivial solution $\mathbf{a} = \mathbf{0}$ for all ω. In the harmonic amplitudes and natural frequency space the primary bifurcation points occur on the axis ω. These points can be determined by solving the linearized eigenvalue problem associated with eqn (4) which can be written in the form

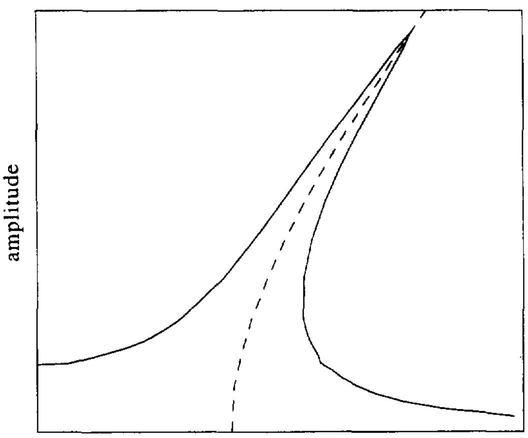

Fig. 1. The typical backbone curve (the dashed line) and the response curve (the solid line).

$$(\mathbf{K}_{lin} - \omega^2 \mathbf{B})\mathbf{a} = \mathbf{0}, \tag{5}$$

where $\mathbf{K}_{lin} = \lceil \alpha_{11}\mathbf{K}_0, \ldots, \alpha_{nn}\mathbf{K}_0 \rceil$.

Taking into account a form of matrices $\mathbf{K}_{lin}$ and $\mathbf{B}$ this problem could be written as a set of independent linear eigenvalue problems of the form

$$(\mathbf{K}_0 - z_l^2 \omega^2 \mathbf{M})\mathbf{a}_l = \mathbf{0}, \tag{6}$$

where $l = 1, 2, \ldots, n$.

It results from eqn (6) that according to the number of harmonics n assumed in the solution of motion equation a greater or smaller set of ω, i.e. the primary bifurcation points has been obtained.

Let us denote by Ω_{lin}, the set of eigenvalues associated with $z_r = 1$. These eigenvalues are simply the linear, natural frequencies of structure. It is easy to notice, that for $z_i \neq 1$ the set of corresponding eigenvalues of eqn (6) for $l = i$ is given by

$$\boldsymbol{\Omega}_i = \frac{1}{z_i}\boldsymbol{\Omega}_{lin} \tag{7}$$

and the corresponding eigenvectors are identical, respectively, as in the case $z_r = 1$ with an accuracy to the constants.

The above-mentioned primary bifurcation points associated with $z_i \neq 1$ indicate the places on axis ω around which the secondary resonances can occur [see a book by Szemplińska-Stupnicka (1990)]. When the primary bifurcation points are known, the backbone curve emanating from the bifurcation points could be traced.

In a case of forced vibrations there are no trivial solutions of matrix amplitude eqn (1) and the one given by $\mathbf{a} = \mathbf{0}$ and λ equal to a prescribed value could be taken as the initial approximation of solution in the numerical continuation procedure. The values of λ must be taken far away from the resonance regions to achieve the convergence conditions of continuation method.

2.2. *Description of the continuation algorithm*

The continuation method is an incremental–iterative one. The solution of matrix amplitude equation is represented by a sequence of frequencies and amplitudes vectors, i.e. ${}^m\mathbf{a}$, ${}^m\lambda$, $m = 1, 2 \ldots$. For any incremental step, the vector ${}^m\mathbf{a}$ and the frequency ${}^m\lambda$ of the preceding step is assumed to be given. The purpose of an incremental method is to find an increment of frequency $\Delta\lambda^2$ and an increment of amplitude vector $\Delta\mathbf{a}$, which can be accumulated to yield

$${}^{m+1}\mathbf{a} = {}^m\mathbf{a} + \Delta\mathbf{a}, \quad {}^{m+1}\lambda^2 = {}^m\lambda^2 + \Delta\lambda^2. \tag{8}$$

Following the continuation method, as described for example by Seydel (1988), the constraints equation is added to the matrix amplitude eqn (1) in the form proposed by Crisfield (1981)

$$g(\Delta\mathbf{a}, \Delta\alpha) = (\mathbf{a} - {}^m\mathbf{a})^t(\mathbf{a} - {}^m\mathbf{a}) = \Delta\mathbf{a}^t\,\Delta\mathbf{a} = \Delta\alpha^2, \tag{9}$$

where $\Delta\alpha$ is the increment of the arc-length α.

The eqns (1) and (9) are a coupled set of equations where $\mathbf{a}$ and λ^2 are the unknown quantities. Because of a nonlinearity of the problem it can be solved only by an iterative procedure. Suppose, after the iteration i, we know some approximation of solution denoted by $\mathbf{a}^i$ and $(\lambda^2)^i$. The iteration change of the frequency increment $\delta\lambda^2$ and the amplitude increment $\delta\mathbf{a}$ are governed by the following equation

$$\mathbf{G}_a(\mathbf{a}^i,(\lambda^2)^i)\,\delta\mathbf{a} = -\mathbf{G}(\mathbf{a}^i,(\lambda^2)^i) - \mathbf{G}_\lambda(\mathbf{a}^i,(\lambda^2)^i)\,\delta\lambda^2 \tag{10}$$

where $\mathbf{G}_a$ is the matrix of first derivatives of vector $\mathbf{G}$ with respect to $\mathbf{a}$ and $\mathbf{G}_\lambda$ is the vector of first derivatives of $\mathbf{G}$ with respect to λ^2. The explicit form for $\mathbf{G}_a$ and $\mathbf{G}_\lambda$ is given in the paper by Lewandowski (1996).

Using the approach proposed by Batoz and Dhatt (1979) the vector $\delta\mathbf{a}$ can be written as a sum of two components

$$\delta\mathbf{a} = \delta\bar{\mathbf{a}} + \delta\tilde{\mathbf{a}}\,\delta\lambda^2, \tag{11}$$

where the first one represents an influence of the residual vector $\mathbf{G}(\mathbf{a}^i,(\lambda^2)^i)$ and the second one is due to $\delta\lambda^2$. The vectors $\delta\bar{\mathbf{a}}$ and $\delta\tilde{\mathbf{a}}$ are determined by the following relations

$$\begin{aligned}\delta\bar{\mathbf{a}} &= -\mathbf{G}_a^{-1}(\mathbf{a}^i,(\lambda^2)^i)\mathbf{G}(\mathbf{a}^i,(\lambda^2)^i),\\ \delta\tilde{\mathbf{a}} &= -\mathbf{G}_a^{-1}(\mathbf{a}^i,(\lambda^2)^i)\mathbf{G}_\lambda(\mathbf{a}^i,(\lambda^2)^i).\end{aligned} \tag{12}$$

Substituting the total increment of $\mathbf{a}$ up to the $(i+1)$th iteration given by $\Delta\mathbf{a}^{i+1} = \Delta\mathbf{a}^i + \delta\mathbf{a}$ into the constraints eqn (9) gives the following equation for $\delta\lambda^2$

$$a_1(\delta\lambda^2)^2 + a_2\,\delta\lambda^2 + a_3 = 0, \tag{13}$$

where

$$\begin{aligned}a_1 &= \delta\tilde{\mathbf{a}}^t\,\delta\tilde{\mathbf{a}},\\ a_2 &= 2(\Delta\mathbf{a}^i + \delta\bar{\mathbf{a}})^t\,\delta\tilde{\mathbf{a}},\\ a_3 &= (\Delta\mathbf{a}^i + \delta\bar{\mathbf{a}})^t(\Delta\mathbf{a}^i + \delta\bar{\mathbf{a}}) - \Delta\alpha^2.\end{aligned} \tag{14}$$

In eqn (13), the increment $\delta\lambda^2$ which gives a positive value of $(\Delta\mathbf{a}^{i+1})^t\,\Delta\mathbf{a}^i$ is taken as the nontrivial solution to avoid doubling back on the response curve. If both solutions give negative or positive values to $(\Delta\mathbf{a}^{i+1})^t\,\Delta\mathbf{a}^i$, the corresponding incremental step is automatically restarted with the arc-length reduced to a half. Also, in the case of negative discriminant of eqn (13), the same procedure is followed. To prevent the number of iterations from being too large, a maximum number of iterations is set. If the number of iterations exceeds that, the incremental step is restarted according to the same procedure as before.

A new approximation of solution of matrix amplitude equation is given by:

$$\begin{aligned}\Delta\mathbf{a}^{i+1} &= \Delta\mathbf{a}^i + \delta\mathbf{a}, \quad (\Delta\lambda^2)^{i+1} = (\Delta\lambda^2)^i + \delta\lambda^2,\\ \mathbf{a}^{i+1} &= \mathbf{a}^i + \Delta\mathbf{a}^i, \quad (\lambda^2)^{i+1} = (\lambda^2)^i + (\Delta\lambda^2)^i.\end{aligned} \tag{15}$$

The iterations are repeated until the following inequalities:

$$|\lambda^{i+1} - \lambda^i| < \varepsilon_1, \quad \|\mathbf{a}^{i+1} - \mathbf{a}^i\| < \varepsilon_2, \|\mathbf{G}(\mathbf{a}^{i+1}, \lambda^{i+1}) - \mathbf{G}(\mathbf{a}^i, \lambda^i)\| < \varepsilon_3, \tag{16}$$

are satisfied, where ε_1, ε_2, ε_3 are the assumed accuracy of calculations.

After calculation of a new point on the response curve given by ${}^{m+1}\lambda$ and ${}^{m+1}\mathbf{a}$ an existence of critical point is checked by determining a sign of matrix $\mathbf{G}_a({}^{m+1}\mathbf{a}, {}^{m+1}\lambda)$ determinant. If for two successive points we have different signs of this determinant then in another point $\hat{\lambda}$, $\hat{\mathbf{a}}$ between previous ones $\det \mathbf{G}_a(\hat{\mathbf{a}}, \hat{\lambda}) = 0$ and a sufficient condition for existence of critical points is fulfilled. The coordinates of this critical point are accurately calculated by using the bisection method. In this case the increment of arc-length is reduced by half and a medial point is calculated. On a basis of sign of the determinant of matrix $\mathbf{G}_a$

in this point a range of existence of critical point is appropriately reduced. In the next section, some properties of critical points are discussed and the procedure for tracing the branches emanating from the secondary bifurcation points is described.

2.3. *Conditions of critical points appearing on the response curve and branching after bifurcation*

In a context of computational mechanics an analysis of stability points is devoted to the conservative systems for which the tangent matrix is symmetric. Among others the noticeable papers in this field are written by Riks (1984), Allman (1989), Kouchia and Mikkola (1989) and by Flores and Godoy (1992). In this work the mentioned analysis is given for the non-conservative systems.

The first derivative of eqn (3) with respect to the arc-length parameter α in the amplitude-frequency space is

$$\frac{\partial \mathbf{G}}{\mathrm{d}\alpha} = \frac{\partial \mathbf{G}}{\partial \mathbf{a}}\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\alpha} + \frac{\partial \mathbf{G}}{\partial \kappa}\frac{\mathrm{d}\kappa}{\mathrm{d}\alpha} = \mathbf{G}_a\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\alpha} + \mathbf{G}_\kappa\frac{\mathrm{d}\kappa}{\mathrm{d}\alpha}, \tag{17}$$

and it must vanish along the response curve. Above, we introduce notation $\kappa = \lambda^2$. The second derivative is related to the curvature of response curve and has the form

$$\frac{\mathrm{d}^2\mathbf{G}}{\mathrm{d}\alpha^2} = \frac{\partial \mathbf{G}}{\partial \mathbf{a}}\frac{\mathrm{d}^2\mathbf{a}}{\mathrm{d}\alpha^2} + \frac{\partial \mathbf{G}}{\partial \kappa}\frac{\mathrm{d}^2\kappa}{\mathrm{d}\alpha^2} + \left(\frac{\partial^2 \mathbf{G}}{\partial \mathbf{a}^2}\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\alpha}\right)\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\alpha} + 2\frac{\partial^2 \mathbf{G}}{\partial \mathbf{a}\,\partial \kappa}\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\alpha}\frac{\mathrm{d}\kappa}{\mathrm{d}\alpha} + \frac{\partial^2 \mathbf{G}}{\partial \kappa^2}\left(\frac{\mathrm{d}\kappa}{\mathrm{d}\alpha}\right)^2, \tag{18}$$

and also vanishes along the response curve.

The solution of eqn (17) with respect to $\mathrm{d}\mathbf{a}/\mathrm{d}\alpha$ is unique as long as the matrix $\mathbf{G}_a$ is not singular. When a critical point is attained, the conditions

$$\mathbf{G}_a\mathbf{q}_r = \mathbf{0}, \quad \mathbf{q}_l^t\mathbf{G}_a = \mathbf{0}, \tag{19}$$

are satisfied, where $\mathbf{q}_r$ and $\mathbf{q}_l$ are the right and left eigenvectors associated with the zero eigenvalue of matrix $\mathbf{G}_a$, respectively. Multiplying the equation $\mathrm{d}\mathbf{G}/\mathrm{d}\alpha = \mathbf{0}$ by $\mathbf{q}_l^t$ and taking into account the relations (17) and (19), the condition for existence of solution of the considered equation is given by

$$\mathbf{q}_l^t\mathbf{G}_\kappa\frac{\mathrm{d}\kappa}{\mathrm{d}\alpha} = \mathbf{0}. \tag{20}$$

Two different types of critical points are defined on a basis of eqn (20):

(1) the bifurcation points in which

$$\mathbf{q}_l^t\mathbf{G}_\kappa = \mathbf{0}, \tag{21}$$

and $\mathrm{d}\kappa/\mathrm{d}\alpha \neq 0$,

(2) whereas in the limit points $\mathrm{d}\kappa/\mathrm{d}\alpha \neq 0$ and

$$\mathbf{q}_l^t\mathbf{G}_\kappa \neq \mathbf{0}. \tag{22}$$

At the limit point, the tangent of response curve and the right eigenvector $\mathbf{q}_r$ coincide because in this case the eqns (19a) and

$$\mathbf{G}_a\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\alpha} = \mathbf{0}, \tag{23}$$

must be fulfilled simultaneously and the continuation procedure described previously can be directly applied to follow a post critical part of response curve. Also, the primary path of response curve passing through the bifurcation point can be traced without difficulties. However, in order to switch to the secondary branch the approximate direction of this branch must be determined. It is assumed, that the direction of the primary path denoted by $\mathbf{da}/d\alpha = \mathbf{t}$ and $d\kappa/d\alpha = 1$ is known. Taking into account that the matrices $[\mathbf{G}_a, \mathbf{G}_\kappa]$ and $\mathbf{G}_a$ have identical ranges the solution of equation $d\mathbf{G}/d\alpha = \mathbf{0}$ is

$$\frac{\mathbf{da}}{ds} = \zeta\mathbf{q}_r + \eta\mathbf{t}, \quad \frac{d\kappa}{ds} = \eta, \tag{24}$$

where ζ and η are the unknown constants. These constants can be determined by taking into account the following solvability condition of equation $d^2\mathbf{G}/d\alpha^2 = \mathbf{0}$ which can be written in the form

$$\mathbf{q}_l' \left\{ \frac{\partial \mathbf{G}}{\partial \mathbf{a}} \frac{d^2\mathbf{a}}{d\alpha^2} + \frac{\partial \mathbf{G}}{\partial \kappa} \frac{d^2\kappa}{d\alpha^2} + \left(\frac{\partial^2 \mathbf{G}}{\partial \mathbf{a}^2} \frac{\mathbf{da}}{d\alpha} \right) \frac{\mathbf{da}}{d\alpha} + 2 \frac{\partial^2 \mathbf{G}}{\partial \mathbf{a}\, \partial \kappa} \frac{\mathbf{da}}{d\alpha} \frac{d\kappa}{d\alpha} + \frac{\partial^2 \mathbf{G}}{\partial \kappa^2} \left(\frac{d\kappa}{d\alpha} \right)^2 \right\} = 0. \tag{25}$$

Substituting the expression (24) into (25) and taking into account the conditions (19) and (22) we obtain

$$a\zeta^2 + 2b\zeta\eta + c\eta^2 = 0, \tag{26}$$

where

$$a = \mathbf{q}_l'(\mathbf{G}_{aa}\mathbf{q}_r)\mathbf{q}_r, \quad b = \mathbf{q}_l'[(\mathbf{G}_{aa}\mathbf{t} + \mathbf{G}_{a\kappa})\mathbf{q}_r],$$
$$c = \mathbf{q}_l'[(\mathbf{G}_{aa}\mathbf{t} + 2\mathbf{G}_{a\kappa})\mathbf{t} + \mathbf{G}_{\kappa\kappa}], \quad d = b^2 - ac. \tag{27}$$

Following Spence and Jepson (1985), the bifurcation points can be further classified as the transcritical bifurcation points if $a \neq 0$ and $d > 0$ and as the pitchfork bifurcation points if $a = 0$ and $b \neq 0$ or as the isola formation points when $d < 0$.

The eqn (26) has been solved by appending a normalization condition. An example condition suggested by Decker and Keller (1980) is

$$\zeta^2 + 2\eta^2 = 1. \tag{28}$$

In the problems of structural mechanics, the constants a, b and c, are usually calculated approximately. A new and interesting method in this context is presented recently by Flores and Godoy (1992).

In many problems the pitchfork bifurcation occurs. In this case $a = 0$ and the direction of the secondary branch from the bifurcation point is given by

$$\frac{\mathbf{da}}{ds} = \zeta\mathbf{q}_r, \quad \frac{d\kappa}{ds} = 0. \tag{29}$$

The calculation of factors a, b and c is very tedious and requires many additional operations. For these reasons and on a basis of above consideration, in this work, simple procedure described below is used. Let now $\Delta\mathbf{a}_t = \mathbf{t}$ denote the tangent vector to the response curve in the bifurcation point and scaled that

$$\Delta\mathbf{a}_t' \Delta\mathbf{a}_t = \Delta\alpha^2. \tag{30}$$

The trial increment of vector $\Delta\mathbf{a}^0$ along the secondary path is given by

$$\Delta\mathbf{a}^0 = \zeta\,\Delta\mathbf{a}_t + \eta\mathbf{q}_r. \tag{31}$$

The vector $\Delta\mathbf{a}^0$ is constructed in this way that it is orthogonal to the primary path

$$\Delta\mathbf{a}_t^{\prime}\,\Delta\mathbf{a}^0 = 0, \tag{32}$$

and from this and eqn (25), it follows that

$$\begin{aligned} \zeta^2\,\Delta\alpha^2 + 2a_1\zeta\eta + a_2\eta^2 &= \Delta\alpha^2, \\ \zeta\,\Delta\alpha^2 + a_1\eta &= 0, \end{aligned} \tag{33}$$

where

$$a_1 = \Delta\mathbf{a}_t^{\prime}\mathbf{q}_r, \quad a_2 = \mathbf{q}_r^{\prime}\mathbf{q}_r. \tag{34}$$

The solution of eqns (33) is given by

$$\eta = \pm\frac{\Delta\alpha^2}{\sqrt{a_2\,\Delta\alpha^2 - a_1^2}}, \quad \zeta = \mp\frac{a_1}{\sqrt{a_2\,\Delta\alpha^2 - a_1^2}}. \tag{35}$$

When the eigenvector $\mathbf{q}_r$ is orthogonal to the primary path $a_1 = 0$ and from relations (31) and (35) one obtains $\zeta = 0$, $\eta \neq 0$ and

$$\Delta\mathbf{a}^0 = \eta\mathbf{q}_r. \tag{36}$$

At the end it is noted that Rheinbolt (1978) has developed some iterative procedure where distinction between different types of bifurcation is not important.

2.4. *Stability analysis of the steady-state solution*

On a basis of Floquet theory the stability analysis of steady state solution is presented in the companion paper by Lewandowski (1996). It is pointed out that the solution of quadratic eigenvalue problem of the form

$$[\mu^2\mathbf{B} + \mu(\mathbf{Z} + 2\lambda\mathbf{E}) + \mathbf{K} - \lambda^2\mathbf{X} + \lambda\mathbf{D}]\mathbf{q} = \mathbf{0}, \tag{37}$$

is required, where μ is the characteristic multiplier of which the real part must be negative for the stable steady state solution. The matrices $\mathbf{B}$, $\mathbf{Z}$, $\mathbf{E}$, $\mathbf{K}$, $\mathbf{X}$ and $\mathbf{D}$ are defined in the previously mentioned paper. A simplified stability analysis can be made on a basis of eigenvalues μ determined from the following linear eigenvalue problem

$$[\mathbf{K} - \lambda^2\mathbf{X} + \lambda(\mathbf{D} + 2\mu\mathbf{E})]\mathbf{q} = \mathbf{0}. \tag{38}$$

From both quadratic and linear eigenvalue problems very close results (if the limit between the stable and unstable solutions is of interest) are obtained so the linear one is preferred as a more attractive one from a numerical point of view. In this way both the Hopf and saddle-node bifurcation points can be determined. If we look only for the saddle-node bifurcation it is possible to use the condition $\mu = 0$ on a limit between the stable and unstable solutions. In this case

$$(\mathbf{K} - \lambda^2\mathbf{X} + \lambda\mathbf{D})\mathbf{q} = \mathbf{0}, \tag{39}$$

and the condition of existence of non-trivial solution of (39) is

$$\det(\mathbf{K}-\lambda^2\mathbf{X}+\lambda\mathbf{D}) = 0, \tag{40}$$

which is most attractive from a numerical point of view because the stability analysis requires only a calculation of the sign of determinant of matrix $(\mathbf{K}-\lambda^2\mathbf{X}+\lambda\mathbf{D})$ during a typical incremental step.

The solution of full eigenvalue spectrum at each step of the continuation process is very expensive and in many instances for the systems with the huge number of degrees of freedom is out of question. Only the evolution of the lowest eigenvalues by the power iteration method or the evolution of the determinant of tangent matrix can be done with reasonable cost. Additional studies on this subject are very desired and any possible simplification must be exploited.

2.5. *Some computational aspects of the proposed continuation procedure*

Starting the incremental procedure the initial increment of arc-length is taken as

$$\Delta\alpha^2 = \mathbf{d}^t\mathbf{d}, \tag{41}$$

where $\mathbf{d}$ is the appropriately scaled eigenvalue vector corresponding to the chosen linear frequency of oscillation or in other words to a chosen primary bifurcation point from which the traced branch emanates. In a case of forced vibration problem $\Delta\alpha$ is taken in the same way but now $\mathbf{d}$ denotes the vector of amplitudes determined by the Newton method for λ far away from the resonance region.

The arc-length for an increment step other than the first one is determined with reference to the number of iterations in the last step, I_i:

$$\Delta\alpha_{i+1}^2 = \Delta\alpha_i^2(I_d/I_i)^{1/2}, \tag{42}$$

where I_d is the desired number of iterations.

In each incremental step an initial approximation of frequency increment $(\Delta\lambda^2)^1$ is given by

$$(\Delta\lambda^2)^1 = \pm\Delta\alpha(\delta\bar{\mathbf{a}}^t\,\delta\bar{\mathbf{a}})^{-1/2}, \tag{43}$$

where a sign in eqn (43) is the same as a sign of total increment of frequency in the previous incremental step. The sign in this relation is chosen arbitrarily if the incremental procedure is started.

3. NUMERICAL EXAMPLES

In this section the proposed method is applied to transversal vibration of multispan beams which ends are restrained from longitudinal movement. The von Karman theory is used to describe the behavior of such systems. The interesting results of numerical calculation are presented and discussed. The two node beam element with two degrees of freedom per node and the cubic shape functions, described in detail by Lewandowski (1991), is used in our own computer program.

3.1. *Simply supported, single span beam*

First, the results obtained by different methods for a simply-supported, single span beam are presented. In a case of free undamped vibration several analytical and numerical solutions are reported in the literature. A comparison of a non-dimensional ratio of first

Table 1. Comparison with published results for a simply supported beam—the solution with one harmonic

	$(\omega/\omega_1)^2$				
α/r	Present method	Rao (1976)	Mei (1972)	Srinivasan (1969)	Woinowsky-Krieger (1950)
1.0	1.1875	1.1855	1.1857	1.1874	1.1864
2.0	1.7500	1.7211	1.7379	1.7477	1.7366
3.0	2.6875	2.5670	2.6439	2.6798	2.6429

Table 2. Comparison of results for solutions with one and two harmonics

	ω/ω_l		Present method (6 elements)			
	Cheung (1982) 1 harmonic solution	Cheung (1982) 2 harmonics solution	1 harmonic solution		2 harmonics solution	
α/r			α/r	ω/ω_l	α/r	ω/ω_l
1.0	1.1029	1.0892	0.9815	1.0865	1.0087	1.0906
2.0	1.3630	1.3178	2.0283	1.3331	1.9738	1.3106
3.0	1.7050	1.6255	3.0089	1.6422	2.9808	1.6198

natural frequency calculated in a different way is presented in Tables 1 and 2. The results of the solution with only one harmonic, $\cos \omega t$, are compared in Table 1, whereas in Table 2 the ones with two harmonics, $\cos \omega t$ and $\cos 3\omega t$, are shown. The symbols ω/ω_l, r and α denote the ratio of first nonlinear natural frequency normalized with respect to the first linear one, the radius of inertia of the beam cross-section and the total amplitude of vibration in the middle of the beam, respectively. All results reported by different authors are very close and it is visible that the differences between the solutions with one and two harmonics are small and that means a faster convergence of Fourier series. Six elements are used in our own calculation.

However, careful examination of bifurcation condition shows that the secondary bifurcation points and an additional branch of backbone curve exist if two harmonics are taken into account in the Fourier series. This problem was previously studied by Lewandowski (1994) in a purely analytical way and by using the finite element method. The complete backbone curve is shown in Fig. 2. As a matter of fact, it is a projection of the space curve onto a plane parameterized by α_1/r and ω/ω_l, where now α_1 denotes the amplitude of the first harmonic in a quarter of beam span. In a considered range of frequency values there are two primary bifurcation points. These points, related to the first linear frequency and the second one divided by three, are shown as the small circles,

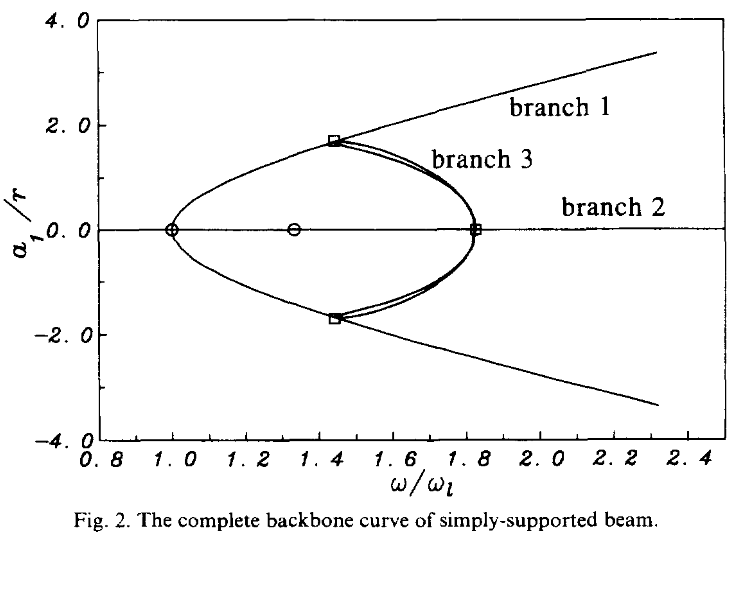

Fig. 2. The complete backbone curve of simply-supported beam.

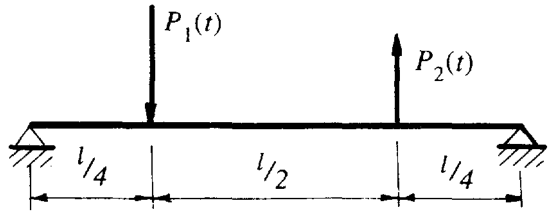

Fig. 3. The simply-supported beam under the excitation forces.

respectively. The secondary bifurcation points are visible as the small squares. The branch 1 is very similar to a classical backbone curve whereas the second one emanating from the second primary bifurcation points coincides in this figure with axis ω/ω_l. These two branches are linked by the third one which intersects both of them in the secondary bifurcation points. Two additional branches indicate the range of parameters where the secondary resonances can occur when the beam is subjected to the excitation forces.

The resonance curve for a beam subjected to the external forces shown in Fig. 3 is also determined. The results are presented in Fig. 4, where the modulus of non-dimensional amplitude of the first harmonic at $x = 0.75l$ vs the nondimensional frequency λ/ω_l is shown. It was assumed that $\mathbf{C} = c\mathbf{M}$, where $c/2\omega_l = 0.005$ and the beam is divided into eight equal elements. Moreover, the excitation forces $P_1(t) = 13.63EJr/l^3 \cos \lambda t$ and $P_2(t) = 9.62EJr/l^3 \cos \lambda t$, where EJ denotes the bending rigidity of beam. Two harmonics are taken into account in the Fourier series so the solution can be written in the form

$$\mathbf{a}(t) = \mathbf{a}_1 \cos \lambda t + \mathbf{b}_1 \sin \lambda t + \mathbf{a}_2 \cos 3\lambda t + \mathbf{b}_2 \sin 3\lambda t. \tag{44}$$

In this case the internal resonance occurs as it is indicated previously by the branch 3 on Fig. 2. The effects of internal resonance are visible in Fig. 4 as the small slopes. However, this phenomenon occurs only if a distribution of excitation forces contains a significant anti-symmetric part with respect to a beam axis of symmetry. The stable parts of response curve are plotted by the solid lines whereas the unstable ones are plotted by the dashed lines. This problem is also solved by using the direct integration method (the Newmark average acceleration method) to calculate the steady-state response and the results are also shown, in Fig. 4, by the small crosses. In this case the equation of motion (1) is numerically integrated from the appropriately chosen initial conditions up to time when the steady-state is reached. As is visible, the results obtained by both methods are very close. However, time of computer calculations of one steady-state solution using the Newmark method is from a few hundreds to a few thousands times greater than time necessary for determination

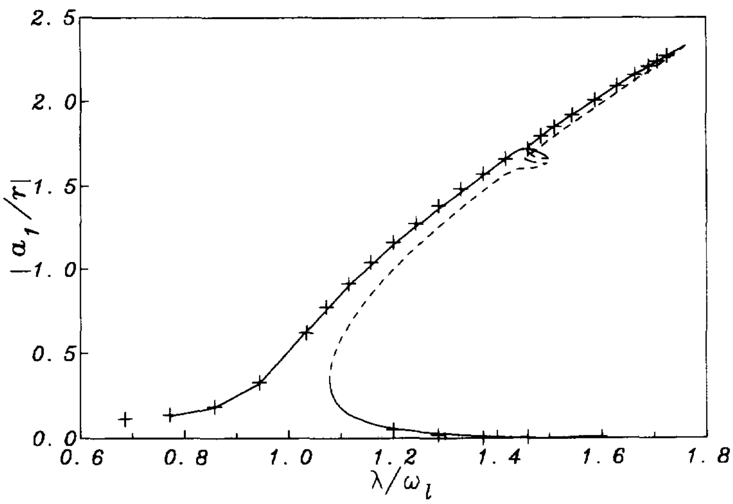

Fig. 4. The resonance curve for the simply-supported beam $c/2\omega = 0.005$.

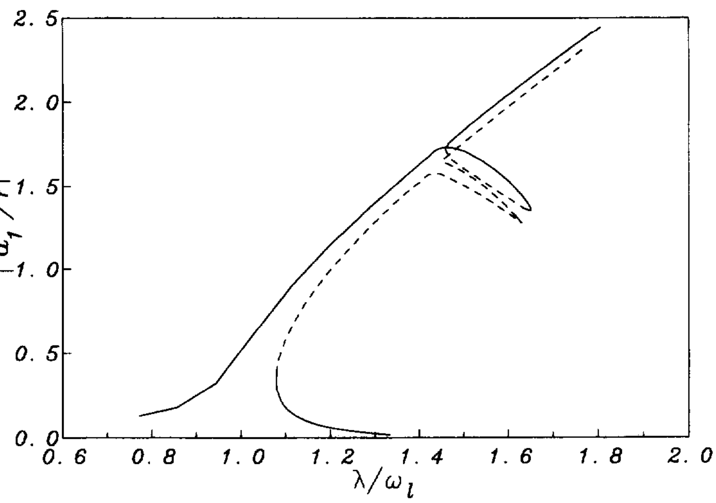

Fig. 5. The resonance curve for the simply-supported beam $c/2\omega = 0.001$.

of the whole response curve by the proposed method. This is a great advantage of the suggested method.

The effects of internal resonances are more significant when damping forces are smaller as it is shown for $c/2\omega = 0.001$ in Fig. 5.

3.2. *Two span beam on elastic support in the middle of the beam*

A two span beam shown in Fig. 6 with equal spans L is considered. The beam is divided into eight equal elements. The support in the middle of the beam is elastic. The characteristic feature of this beam is that for the non-dimensional support ratios kL^3/EJ taken from the range $135 < kL^3/EJ < 800$ the backbone curves "cross" themselves in the classical solution with only one harmonic taken into account. The backbone curves shown in Fig. 7 are obtained if two harmonics are present in the assumed solution of motion equation, i.e.:

$$\mathbf{a}(t) = \mathbf{a}_1 \cos\omega t + \mathbf{a}_2 \cos 3\omega t, \tag{45}$$

where the amplitude a_1 in this figure denotes a component of vector $\mathbf{a}_1$ which is the amplitude of first harmonic in the middle of first span. As previously, the primary and secondary bifurcation points are pictured by the small circles and squares, respectively. In the described picture the curves labeled by the numbers 1 and 2 are very similar to the ones obtained assuming the one harmonic solution, i.e. $\mathbf{a}_2 = \mathbf{0}$. The influence of third harmonic on solution (45) is very small. Also the additional periodic solution exists and this is pictured by the presence of the curve 3 in Fig. 7. Generally speaking, this solution contains two linear modes of vibration (the first and the second ones) in the vector $\mathbf{a}_1$, or from a different point of view, the vector $\mathbf{a}_1$ can be very accurately expanded as the series of eigenvectors using only two linear modes of vibration. For different total amplitudes of vibration there are many possible combinations in which these modes can mutually interact. An important feature of considered solution is that it exists only for one particular value of frequency

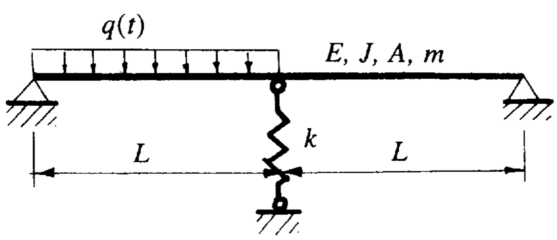

Fig. 6. The two span beam on the elastic support.

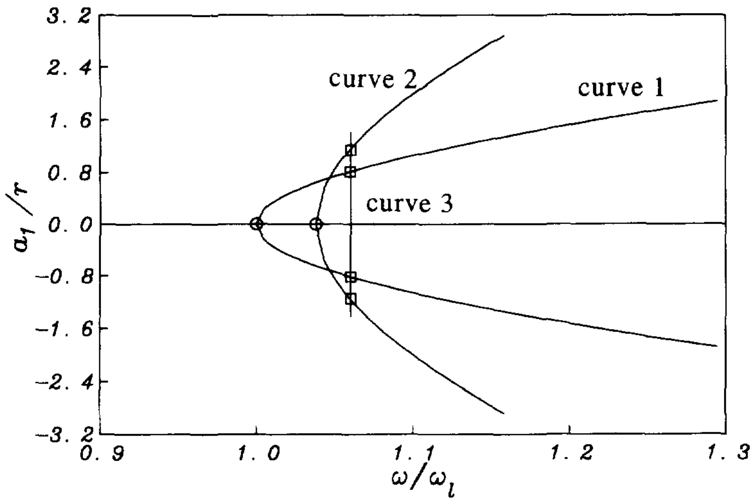

Fig. 7. The backbone curves for the two span beam.

ratio equal to 1.0603. As for the usually determined backbone curves (the curves 1 and 2 in Fig. 7) the elements of vector $\mathbf{a}_2$ are very small in comparison with elements of vector $\mathbf{a}_1$. The aforementioned solution can be also determined by using the one harmonic solution with a little less accuracy. In detail, this particular problem is recently analyzed using a purely analytical method in the paper by Lewandowski (1996). In the author's opinion, the curve 3 indicates the range of parameters in which the internal resonance phenomenon of type 1 : 1 can occur if the beam is excited by the external forces.

In almost all works concerning the free vibration problems of nonlinear systems only one harmonic is taken into account in the solution of motion equations. Moreover, very often the vector $\mathbf{a}_1$ is assumed to be proportional to a chosen linear mode of vibration. In Fig. 8 a comparison of the linear modes of vibration and the deformed shapes of beam resulting from the vector $\mathbf{a}_1$ is presented. The first, anti-symmetric linear and "nonlinear" modes of vibration are identical for all amplitudes of vibration. The backbone curve associated with this mode is described in Fig. 7 as the curve 1. However, there are some differences between the second linear and "nonlinear" modes of vibration. This is shown in Fig. 8 for $a/r = 3.084$ where a denotes the value of "nonlinear" mode in the middle of first span. In Fig. 7, the curve 2 is the one associated with this mode. As mentioned above the vector $\mathbf{a}_1$ associated with curve 3 in Fig. 7 can be understood as a sum of two linear

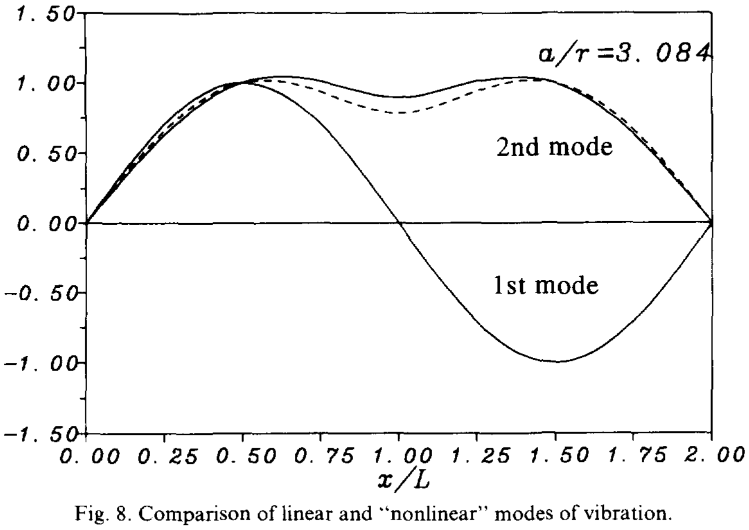

Fig. 8. Comparison of linear and "nonlinear" modes of vibration.

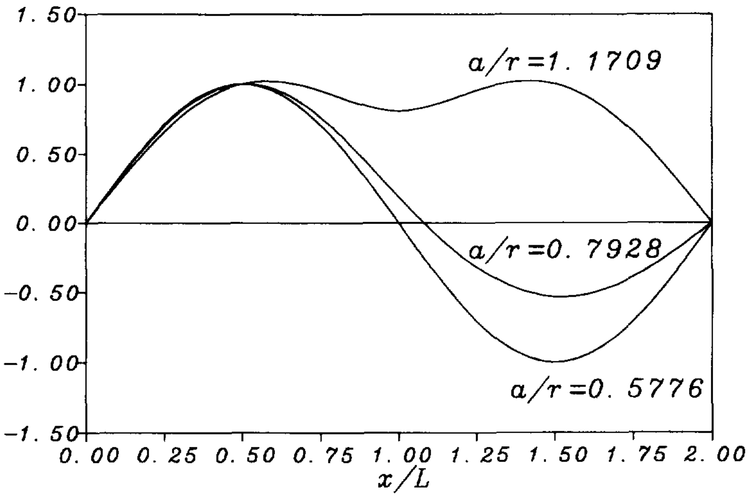

Fig. 9. The shape of "nonlinear" mode of vibration associated with the curve 3 in Fig. 7.

modes of vibration. This is visible in Fig. 9 where the shapes of deflected beam resulting from vector $\mathbf{a}_1$ are shown for the different non-dimensional amplitudes a/r.

The steady-state vibration of the considered beam excited by the uniformly distributed excitation forces $p = q/mr\omega_l^2$ is also analyzed. The damping matrix is $\mathbf{C} = c\mathbf{M}$, where $c/2\omega_l = 0.01$ and the solution is assumed in the form

$$\mathbf{a}(t) = \mathbf{a}_1 \cos \lambda t + \mathbf{b}_1 \sin \lambda t. \tag{46}$$

In Fig. 10 the response curve for $p = 0.05$ is shown. The non-dimensional amplitude a/r ($a^2 = a_1^2 + b_1^2$) is the one in the middle of first span. The unstable part of response curve is shown as the dashed line and, for convenience, the backbone curves are visible as the dotted lines. Also, the results obtained by the Newmark method are marked by small crosses. A good agreement between results of both methods is achieved. Moreover, it was found by both methods that between the frequency ratio λ/ω_l 1.09 and 1.115 the almost periodic steady-states exist and the periodic ones are unstable. This region is small and cannot be clearly shown in Fig. 10. In this region, the stability analysis gives us two characteristic exponents μ which are the complex numbers with positive real parts.

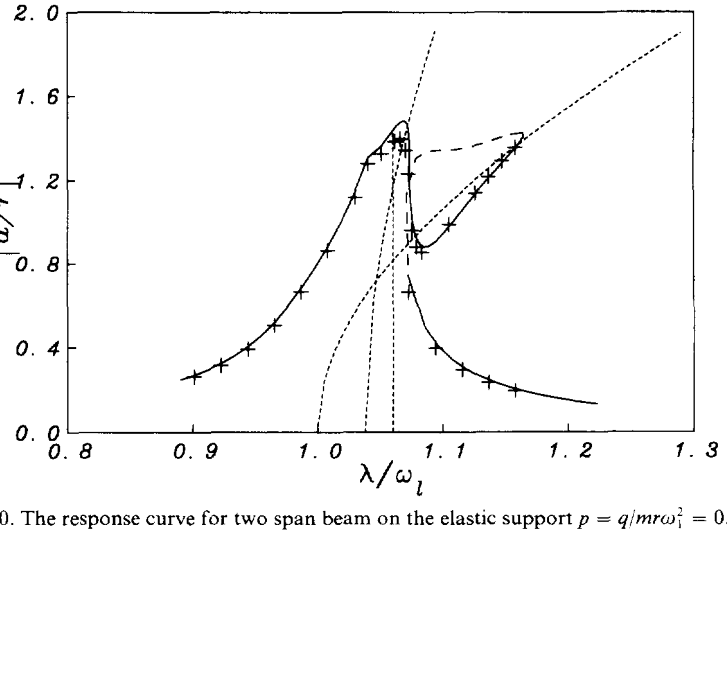

Fig. 10. The response curve for two span beam on the elastic support $p = q/mr\omega_l^2 = 0.05$.

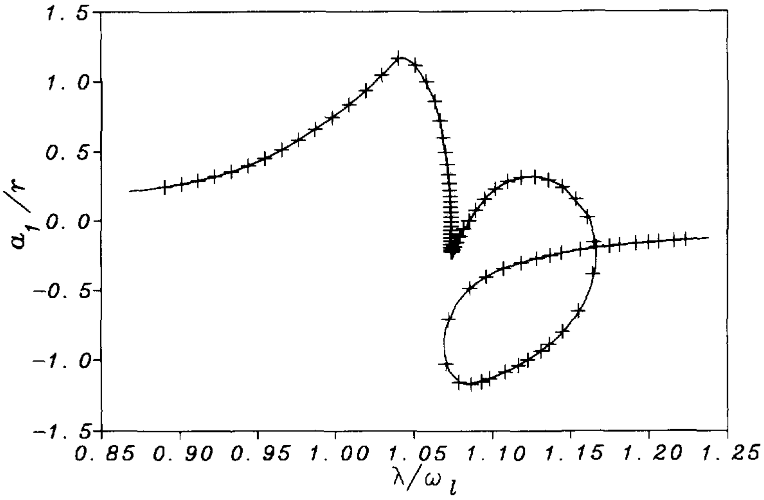

Fig. 11. Comparison of non-dimensional amplitude a_1/r for the one and two harmonics solutions.

The two harmonics solution in the form given by eqn (44) is also calculated and it is found that the influence of 3λ harmonic is small. It is shown in Fig. 11 where a comparison of non-dimensional amplitude a_1/r resulting from the one harmonic solution (the solid line) and the two harmonics solution (the small crosses) are presented. In this figure, a_1 denotes the amplitude of first harmonic in the middle of first span. A complicated shape of curve in Fig. 11 illustrates also a power of the proposed procedure for solving the matrix amplitude equation with parameter.

The effects of previously suggested internal resonances are not very significant in a case presented in Fig. 10. However, the internal resonance phenomenon is much more important if $p = 0.041$, $c/2\omega_l = 0.0064$. The results for this case are presented in Fig. 12. The additional branch of response curve is visible as in Fig. 7. The region of almost periodic vibration is larger than in the previous case and marked in Fig. 12 by the small squares.

Another example of very unusual backbone curves is plotted in Fig. 13. These are the results obtained for the non-dimensional support stiffness $kL^3/EJ = 240$. The beam is divided into 16 equal elements. Two harmonics are taken into account in the assumed solution given by relation (45). The primary bifurcation points numbered by 1 and 2 are related to the first and second linear frequency, respectively, whereas the primary bifurcation point 3 is related to the fourth linear frequency divided by three. The curves 1 and 2 are very similar to the classical backbone curves obtained by assuming the one harmonic solution. The curve emanating from point 3 has all elements of vector $\mathbf{a}_1$ equal to zero and

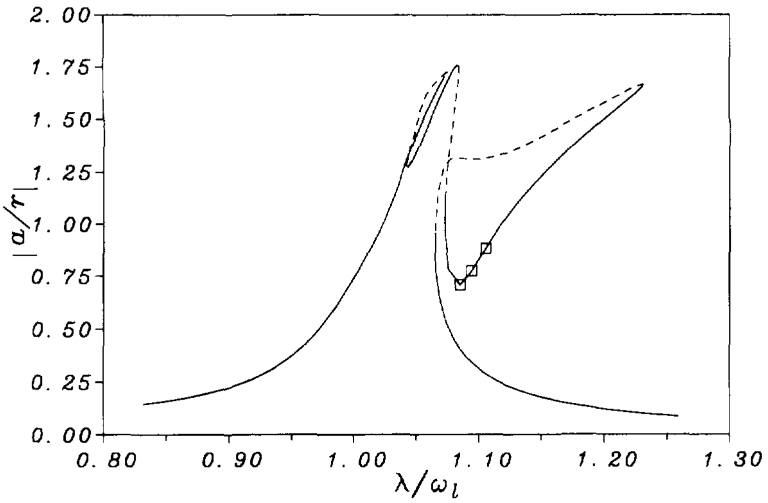

Fig. 12. The response curve for the two span beam. $p = 0.041$.

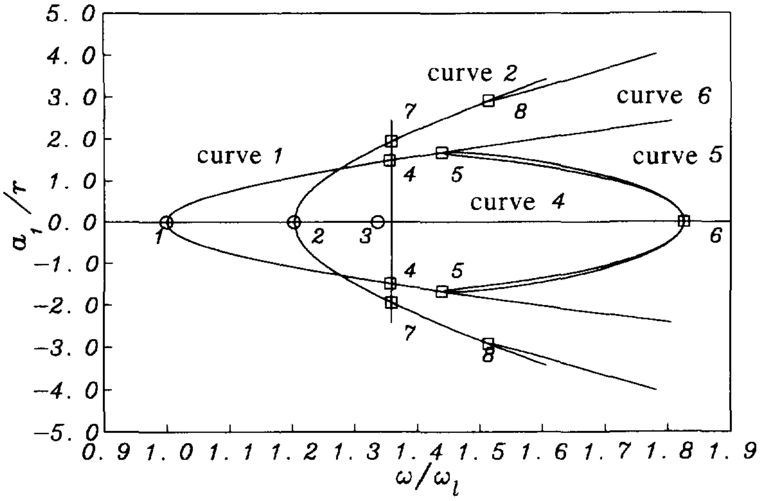

Fig. 13. The backbone curve for two span beam on the elastic support and $kL^3/EK = 240$.

for this reason is not shown in Fig. 13. On these curves many secondary bifurcation points numbered from 4 to 8 exist. In these points three additional branches intersect with the previously described classical backbone curves. The additional branch (the curve 4) is similar to the curve 3 in Fig. 7 and the curve 5 looks like the curve 3 in Fig. 2. Both curves indicate that in this case the internal resonance of type 1 : 1 (the curve 4) and 1 : 3 (the curve 5) can occur.

4. CONCLUSIONS

In this paper, the numerical strategy for solving a set of algebraic equations describing the free and steady-state vibration of geometrically nonlinear structures is presented. The continuation method is used so the strongly nonlinear system of equations with parameter can be solved with an appropriate accuracy and the complicated response curves can be constructed. In the proposed method, the solutions with many harmonics are taken into account which means that the main and secondary resonances can be analyzed in a uniform way. Simple but carefully chosen examples are calculated and the results show that the proposed method and the numerical strategy is effective, efficient and accurate. All of these make the proposed method general, complete and attractive from a computational point of view.

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